to the action of electricity by means of wires introduced through the stoppers.

By immersing about fifty substances in the liquid acid for various periods of time, he has found that it is comparatively a chemically inert substance, and not deoxidized by any ordinary deoxidizing agent except the alkali-metals. Its solvent power is extremely limited; it dissolves camphor freely, iodine sparingly, and a few other bodies in small quantities; it does not dissolve oxygen-salts, and it does not redden solid extract of litmus; it penetrates gutta percha, dissolves out the dark-brown colouring matter, and leaves the gutta percha undissolved, and much more white. It also acts in a singular and somewhat similar manner upon india-rubber; the india-rubber whilst in the liquid acid exhibits no change, but immediately on being taken out it swells to at least six or eight times its original dimensions, and then slowly contracts to its original volume, evidently from expansion and liberation of absorbed carbonic acid; and it is found to be perfectly white throughout its substance. These effects upon gutta percha and india-rubber may prove useful for practical purposes.

The liquid acid is a strong insulator of electricity; sparks (from a Ruhmkorff's coil) which would pass readily through $\frac{9}{32}$nds of an inch of cold air, would with difficulty pass through about $\frac{1}{70}$th of an inch of the liquid acid.

In its general properties it is somewhat analogous to bisulphide of carbon, but it possesses much less solvent power over fatty substances.

January 31, 1861.

Major-General SABINE, R.A., Treasurer and Vice-President, in the Chair.

Sir William Jardine, Bart., was admitted into the Society.

The following communications were read:—

I. "On Systems of Linear Indeterminate Equations and Congruences." By H. J. STEPHEN SMITH, Esq., M.A., Fellow and Mathematical Lecturer of Balliol College, Oxford. Communicated by Professor J. J. SYLVESTER. Received January 17, 1861.

(Abstract.)

The present communication relates to the theory of the solution, in positive and negative integral numbers, of systems of linear indeterminate equations, having integral coefficients. In connexion with this theory, a solution is also given of certain problems relating to rectangular matrices, composed of integral numbers, which are of frequent use in the higher arithmetic. Of this kind are the two following:—

1. "Given (in integral numbers) the values of the determinants of any rectangular matrix of given dimensions, to find all the matrices, the constituents of which are integers, and the determinants of which have those given values.

2. "Given any rectangular matrix, the determinants of which have a given number D for their greatest common divisor, to find all the supplementary matrices, which, with the given matrix, form square matrices, of which the determinant is D."

A solution of particular, but still very important cases of these two problems, has been already given by M. Hermite. The method by which in this paper their general solution has been obtained, depends on an elementary, but apparently fertile principle in the theory of indeterminate linear systems; viz. that if m be the *index of indeterminateness* of such a system (*i. e.* the excess of the number of indeterminates above the number of really independent equations), it is always possible to assign a set of m solutions, such that the determinants of the matrix formed by them shall admit of no common divisor but unity.

Such a set of solutions is termed a *fundamental* set, and possesses the characteristic property, that every other solution of the system can be integrally expressed by means of the solutions contained in it. A set of *independent* solutions is one in which the determinants of the matrix have a finite common divisor, *i. e.* are not all zero.

The theory of independent and fundamental sets of solutions in some respects resembles that of independent and fundamental systems of units in Lejeune Dirichlet's celebrated generalization of the solution of the Pellian equation.

By the aid of the same principle of fundamental sets, the following criterion is obtained for the resolubility or irresolubility of indeterminate linear systems.

"A linear system is or is not resoluble in integral numbers, according as the greatest common divisor of the determinants of the matrix of the system is or is not equal to the corresponding greatest common divisor of its *augmented* matrix."

[The matrix of a linear system of equations is, of course, the rectangular matrix formed by the coefficients of the indeterminates; the *augmented* matrix is the matrix derived from that matrix, by adding to it a vertical column composed of the absolute terms of the equations.]

A system of linear congruences may, of course, be regarded as a system of linear indeterminate equations of a particular form; and the criterion for its resolubility or irresolubility is implicitly contained in that just given for any indeterminate system. But this criterion may be expressed in a form in which its relation to the modulus is very clearly seen.

Let

$$A_{i,1}x_1+A_{i,2}x_2+\ldots+A_{i,n}x_n\equiv A_{i,n+1},\ \text{mod M},\ i=1,2,3,\ldots n$$

represent a system of congruences; let us denote by ∇_n, ∇_{n-1}, ... ∇_1, ∇_0, the greatest common divisors of the determinant, first minors, &c., of the matrix of the system [so that, in fact, ∇_n is the determinant itself, ∇_1 the greatest common divisor of the coefficients $A_{i,j}$, and $\nabla_0=1$]; by D_n, D_{n-1}, ... D_1, D_0 the corresponding numbers for the *augmented* matrix; let also δ_i and d_i respectively represent the greatest common divisors of M with $\dfrac{\nabla_i}{\nabla_{i-1}}$, and of M with $\dfrac{D_i}{D_{i-1}}$; and put

$$m=d_n\times d_{n-1}\times\ \ldots\ \times d_1,$$
$$\mu=\delta_n\times\delta_{n-1}\times\ \ldots\ \times\delta_1,\ \ldots$$

Then the necessary and sufficient condition for the resolubility of the system is

$$m=\mu\,;$$

and when this condition is satisfied, the number of solutions is precisely m.

The demonstration of this result (which seems to exhaust the theory of these systems) is obtained by means of the following theorem:—

"If $\|A\|$ represent any square matrix in integral numbers, ∇_n its determinant, $\nabla_{n-1}, \nabla_{n-2}, \ldots \nabla_1, \nabla_0$ the greatest common divisors of its successive orders of minors, it is always possible to assign two unit-matrices $\|\alpha\|$ and $\|\beta\|$, of the same dimensions as $\|A\|$, and satisfying the equation

$$\|A\| = \|\alpha\| \times \left\| \begin{matrix} \frac{\nabla_n}{\nabla_{n-1}}, & 0, & 0, & \ldots & 0 \\ 0, & \frac{\nabla_{n-1}}{\nabla_{n-2}}, & 0, & \ldots & 0 \\ 0, & 0, & \frac{\nabla_{n-2}}{\nabla_{n-3}}, & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0, & 0, & \cdot & \cdot & \frac{\nabla_1}{\nabla_0} \end{matrix} \right\| \times \|\beta\|."$$

The following result (among many which may be deduced from this transformation of a square matrix) admits of frequent applications:—

"If D be the greatest common divisor of the determinants of the matrix of any system of n independent linear equations; of the D^n sets of values (incongruous mod. D) that may be attributed to the absolute terms of the equations, the system is resoluble for D^{n-1}, and irresoluble for $D^{n-1}\,(D-1)$."

As an example of the use that may be made of this result, it is shown, in conclusion, that it supplies an immediate demonstration of a fundamental principle in the general theory of complex integral numbers, composed of the root of any irreducible equation, having its first coefficient unity, and all its coefficients integral; viz. that the number of incongruous residues, for any modulus, is always represented by the norm of the modulus. A demonstration of this principle has, however, already been given in the 'Quarterly Journal of Pure and Applied Mathematics,' in a paper signed *Lanavicensis*; to whom, therefore, the honour of priority in this inquiry is due.